Lecture 4

Andrei Antonenko

February 05, 2003

1 Analysis of the system

Now, we can formulate the main theoretical result about the system. This theoretical result follows directly from our practical method of solving them.

Case 1 If during our procedures we got an equation of the form $0 = b$, where $b \neq 0$, then we should stop and deduce that the system has no solution.

Case 2 If in the row echelon form we have no equation of the form $0 = b$, where $b \neq 0$, and we have free variables, than the system has infinitely many solutions, since we can assign to them any (real) values.

Case 3 If in the row echelon form we have no equation of the form $0 = b$, where $b \neq 0$, and we don't have free variables, than the solution can be determined uniquely, so the corresponding system has only 1 solution.

Theorem 1.1. *The system of linear equations can have 0, 1 or infinitely many solutions.*

Last lecture we provided an example of a linear system which had free variables in its REF. Now we will provide an example of a system without free variables, so it would have a unique numerical solution.

Example 1.2. *Let's reduce the following system to the row echelon form and then solve it.*

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - x_2 + x_3 = 3 \end{cases} \tag{1}$$

First variable with nonzero coefficient is x_1, and the first equation has a nonzero coefficient before it too. So, we don't have to interchange the equations. Now, we'll subtract the first *equation from the second one, then multiply it by 2 and subtract from the third one. We'll get the following system (we should not omit the first equation!!!):*

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ \quad x_2 - 2x_3 = -3 \\ \quad - 3x_2 - x_3 = -5 \end{cases} \tag{2}$$

Now, we'll apply the same steps to equations from the 2nd to the 3rd. So, we multiply the second equation by 3 and add it to the third one (again, do not omit any equations!!!):

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ \quad x_2 - 2x_3 = -3 \\ \quad - 7x_3 = -14 \end{cases} \tag{3}$$

Now we have a system in a row echelon form. Now it possible to solve it using the back substitution process. From the last equation we can determine the value for x_3: $x_3 = (-14)/(-7) = 2$. Now, substituting the value for x_3 to the second equation we get:

$$x_2 - 2 \cdot 2 = -3, \quad \textit{and thus} \quad x_2 = -3 + 4 = 1.$$

Now, substituting the values for x_2 and x_3 to the first equation we get:

$$x_1 + 1 + 2 = 4, \quad \textit{and thus} \quad x_1 = 4 - 3 = 1.$$

So, we see that this system has a unique solution $(1, 1, 2)$.

2 Reduced Row Echelon Form

Last time we learned how to solve linear systems. We used the following 2 steps:

1. Reduce the system to an equivalent system in row echelon form by Gaussian Elimination

2. Solve the system in row echelon form using back substitution

Today we'll give another method which can be used instead of Step 2 of the algorithm. To do it we'll consider another interesting special case of systems — systems in reduced row echelon forms.

Definition 2.1. *The system is said to be in* **reduced row echelon form (RREF)** *if*

1. *It is in row echelon form*

2. *All coefficients strictly below and above leading terms of equations are zeros*

3. All leading terms have coefficients equal to 1

Example 2.2. *The following system is in reduced row echelon form:*

$$\begin{cases} x_1 \quad\quad + \ x_3 \quad\quad = 2 \\ \quad\quad x_2 + 2x_3 \quad\quad = 4 \\ \quad\quad\quad\quad\quad\quad x_4 = -2 \end{cases}$$

and the following system is not, since not all of the coefficients above x_4 in the last equation are equal to 0:

$$\begin{cases} x_1 \quad\quad + \ x_3 + 5x_4 = 2 \\ \quad\quad x_2 + 2x_3 \quad\quad = 4 \\ \quad\quad\quad\quad\quad\quad x_4 = -2 \end{cases}$$

The system

$$\begin{cases} -x_1 \quad\quad + \ x_3 \quad\quad = 2 \\ \quad\quad 2x_2 + 2x_3 \quad\quad = 4 \\ \quad\quad\quad\quad\quad\quad x_4 = -2 \end{cases}$$

is not in RREF since the coefficients before x_1 in the 1st equation and x_2 in the 2nd equation are not equal to 1.

Now we'll learn how to solve a system in RREF.

Actually, solving of system in RREF is really easy — we have to figure out which variables are leading and which variables are free, give arbitrary values k_i to free variables, and isolate terms with leading variables. This will be a solution of the system in RREF.

Example 2.3. *Let's solve the following system:*

$$\begin{cases} x_1 \quad\quad + \ x_3 \quad\quad = 2 \\ \quad\quad x_2 + 2x_3 \quad\quad = 4 \\ \quad\quad\quad\quad\quad\quad x_4 = -2 \end{cases}$$

Leading variables are x_1, x_2 and x_4, free variable is x_3. So, we'll assign arbitrary value to x_3:

$$x_3 = k_3$$

and move the terms with it to the right hand side:

$$\begin{cases} x_1 = 2 - k_3 \\ x_2 = 4 - 2k_3 \\ x_4 = -2 \end{cases}$$

So, the solution of this system is

$$\{(2 - k_3;\ 4 - 2k_3;\ k_3;\ -2)\,|k_3 \in \mathbb{R}\}$$

As we can see, solving the system in RREF is really easy — we should not even perform any computations — it's possible simply to write the solution directly from the system!!! But soon we will develop an algorithm which will allow us to transpose any system to its RREF.

Let's consider the system in REF and suppose it doesn't have zero equations. (Otherwise, if it has an equation of the form $0x_1 + \cdots + 0x_n = b$, $b \neq 0$ then the system has no solutions, and if we have an equation of the form $0x_1 + \cdots + 0x_n = 0$, then we can simply omit it). Using the following algorithm we will be able to transpose this system to RREF.

Backward Elimination

1. Divide all of the equations by corresponding numbers to make leading coefficients equal to 1.

2. Let x_i is a leading variable of the last equation. Now subtract the last equation from all the previous equations to make coefficients before x_i in all equations but the last one equal to 0.

3. Apply these steps to all equations but the last one.

The description of this algorithm can look scary, but it is really easy to apply.

Example 2.4. *Last lecture we reduced the system to the row echelon form and than solved it. Now we will try to reduce it to RREF. The row echelon form of this system was:*

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 - x_4 = 2 \\ -x_4 = 5 \end{cases}$$

First of all we'll multiply all the equations by corresponding numbers to make leading coefficients equal to 1. We'll have the following system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 - x_4 = 2 \\ x_4 = -5 \end{cases}$$

Now, we're eliminating x_4-terms in the first 2 equations by adding the third one to the second one:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 = -3 \\ x_4 = -5 \end{cases}$$

Now we need to eliminate the term with x_2 in the first equation. To do this, we're subtracting the second equation multiplied by 2 from the first equation.

$$\begin{cases} x_1 + \quad - x_3 = 8 \\ x_2 + x_3 = -3 \\ x_4 = -5 \end{cases}$$

Now, we have the RREF of our system and we can write the solution to it:

$$\{(8+x_3;\ -3-x_3;\ x_3;\ -5) \,|\, x_3 \in \mathbb{R}\}$$

Example 2.5. *We will provide another example of transforming the system to RREF using the system from the example from the beginning of this lecture. The row echelon form of this system was:*

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ \quad\;\; x_2 - 2x_3 = -3 \\ \qquad\quad - 7x_3 = -14 \end{cases} \tag{4}$$

First of all we'll multiply all the equations by corresponding numbers to make leading coefficients equal to 1. We'll have the following system:

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ \quad\;\; x_2 - 2x_3 = -3 \\ \qquad\qquad x_3 = 2 \end{cases}$$

Now, we're eliminating x_3-terms in the first 2 equations by subtracting the third one from the second one and by adding the third equation multiplied by 2 to the 2nd one:

$$\begin{cases} x_1 + x_2 = 2 \\ \quad\;\; x_2 = 1 \\ \qquad\quad x_3 = 2 \end{cases}$$

Now we need to eliminate the term with x_2 in the first equation. To do this, we're subtracting the second equation from the first equation.

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

This is a RREF of our initial system. The thing we can notice here, is that if the system has a unique solution, the RREF coincides with it.

Actually, the RREF for the system of equation can be defined uniquely, and we can get it just knowing the solution for the system.

Example 2.6. *Let's suppose that we know, that the system has the following solution set:*

$$\{(3+4x_2-x_5;\ x_2;\ 10-2x_5;\ 5;\ x_5) \,|\, x_3,\, x_5 \in \mathbb{R}\}$$

Than we know, that there will be 2 free variables — x_3 and x_5, and 3 equations in a RREF — one for each leading variable x_1, x_3 and x_4. Thus it's easy to write the system:

$$\begin{cases} x_1 - 4x_2 + x_5 = 3 \\ \qquad\quad x_3 + 2x_5 = 10 \\ \qquad\qquad x_4 = 5 \end{cases}$$

Now, we're ready to provide an algorithm for solving systems.

1. Use Gaussian Elimination to reduce the system to a Row Echelon Form

2. Use Backward Elimination to reduce new system in REF to it's Reduced Row Echelon Form

3. Formally write the solution of the system in RREF.

3 Overview of linear systems

So, we developed a method of solving linear systems, and proved our main result about the number of solutions of a linear system. We can summarize our algorithms in the following diagram: